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:

1 $\forall X_0 \in \mathbb{R}^n \ \ \forall \delta > 0$

$$\begin{aligned}
\mathrm{Int} B(X_0\delta) &= B(X_0\delta) \\
\mathrm{Ext} B(X_0\delta) &= \{X \in \mathbb{R}^n \mid \|X - X_0\| > \delta\} \\
\partial B(X_0\delta) &= \{X \in \mathbb{R}^n \mid \|X - X_0\| = \delta\} \\
\overline{B(X_0\delta)} &= \{X \in \mathbb{R}^n \mid \|X - X_0\| \leq \delta\}
\end{aligned}$$

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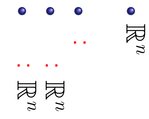
- : $\mathbb{R}^n$ Cauchy
- $\mathbb{R}^n$
- $\mathbb{R}^n$ Cauchy Cauchy
- $\mathbb{R}^n$ $\mathbb{R}^n$ Cauchy


$\mathbb{R}^n$
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$\mathbb{R}^n$

§2 n n

: XY f $x \in X$ Y $y($ $y = f(x))$ $f\,X\,Y$ ( ) $y\,x$ $x\,y$ $f\colon X \to Y$

1 $mn \geq 1$ $\Omega \subseteq \mathbb{R}^n$

$\boldsymbol{f}\colon \Omega \to \mathbb{R}^m$ Ω n $m = 1$ n ( )

- : $\boldsymbol{f}\boldsymbol{g} : \Omega \to \mathbb{R}^m$ $\lambda\mu \in \mathbb{R}\ \forall X \in \Omega$
  $(\lambda \boldsymbol{f} + \mu \boldsymbol{g})(X) = \lambda \boldsymbol{f}(X) + \mu \boldsymbol{g}(X)$
- : $\boldsymbol{f} : \Omega \to \mathbb{R}^m$ $g : \Omega \to \mathbb{R}$ $\forall X \in \Omega$ $(g\boldsymbol{f})(X) := g(X)\boldsymbol{f}(X)$
  $(\frac{\boldsymbol{f}}{g})(X) := \frac{\boldsymbol{f}(X)}{g(X)}(\ g(X) \neq 0)$

- : $lmn \geq 1$ $\Omega_1 \subseteq \mathbb{R}^n$
  $\Omega_2 \subseteq \mathbb{R}^m$ $\boldsymbol{f}\colon \Omega_1 \to \Omega_2$ $\boldsymbol{g}\colon \Omega_2 \to \mathbb{R}^l$
  $\forall X \in \Omega_1 \quad (\boldsymbol{g} \circ \boldsymbol{f})(X) := \boldsymbol{g}(\boldsymbol{f}(X))$
- : $\boldsymbol{f}\colon \Omega \to \mathbb{R}^m$ n $\forall X = (x_1 \dots x_n) \in \Omega$ $\boldsymbol{f}(X) \in \mathbb{R}^m$ $Y = (y_1 \dots y_m)^T$ y_j X : $y_j = f_j(X) = f_j(x_1 \dots x_n)$ $\boldsymbol{f}\colon \Omega \to \mathbb{R}^m$ m n $f_j : \Omega \to \mathbb{R}$ $\boldsymbol{f} = (f_1 \dots f_m)^T$

§3

1 $mn \geq 1$ $\Omega \subseteq \mathbb{R}^n$

$X_0 \in \mathbb{R}^n$ Ω $\boldsymbol{f}\colon \Omega \to \mathbb{R}^m$ $A \in \mathbb{R}^m$ $\forall \varepsilon > 0$ $\exists \delta > 0$ $\forall X \in \Omega$

$0 < \|X - X_0\|_n < \delta$ $\|\boldsymbol{f}(X) - A\|_m < \varepsilon$

X Ω X_0 $\boldsymbol{f}(X)$ A

(A) $\lim\limits_{\Omega \ni X \to X_0} \boldsymbol{f}(X) = A$

- $\lim\limits_{\Omega \ni X \to X_0} \mathbf{f}(X) = A \quad \forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \forall X \in \mathring{B}(X_0 \delta) \cap \Omega \quad \mathbf{f}(X)(A\varepsilon)$
- $\mathbf{f} = (f_1 \dots f_m)^T \; A = (a_1 \dots a_m)^T \quad \lim\limits_{\Omega \ni X \to X_0} \mathbf{f}(X) = A \qquad 1 \leq j \leq m \quad \lim\limits_{\Omega \ni X \to X_0} f_j(X) = a_j$
- $X_0 \; \Omega \cup \{X_0\}$
  $\lim\limits_{\Omega \ni X \to X_0} \mathbf{f}(X) \quad \lim\limits_{X \to X_0} \mathbf{f}(X)$

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-
- : $fg : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ $\lambda\mu \in \mathbb{R}$ $\lim\limits_{\Omega \ni X \to X_0} f(X) = A$ $\lim\limits_{\Omega \ni X \to X_0} g(X) = B$

(a) $\lim\limits_{\Omega \ni X \to X_0} (\lambda f + \mu g)(X) = \lambda A + \mu B$

(b) $\lim\limits_{\Omega \ni X \to X_0} (fg)(X) = AB$

(c) $\lim\limits_{\Omega \ni X \to X_0} \frac{f}{g}(X) = \frac{A}{B}(\ B \neq 0)$

- : $lmn \geq 1$ $\Omega_1 \subseteq \mathbb{R}^n$ $\Omega_2 \subseteq \mathbb{R}^m$ $\boldsymbol{f}\colon \Omega_1 \to \Omega_2$ $\boldsymbol{g}: \Omega_2 \to \mathbb{R}^l$ $\lim\limits_{\Omega_1 \ni X \to X_0} \boldsymbol{f}(X) = Y_0$ $\boldsymbol{f}$ X_0 Y_0 $\lim\limits_{\Omega_2 \ni Y \to Y_0} \boldsymbol{g}(Y) = A$

  $$\lim_{\Omega_1 \ni X \to X_0} (\boldsymbol{g} \circ \boldsymbol{f})(X) = A$$

  :

- : $mn \geq 1$ $\Omega \subseteq \mathbb{R}^n$ $\boldsymbol{f}\colon \Omega \to \mathbb{R}^m$ $X_0 \in \mathbb{R}^n$ $A \in \mathbb{R}^m$ $\lim\limits_{\Omega \ni X \to X_0} \boldsymbol{f}(X) = A$ $\Omega \setminus \{X_0\}$ X_0 $\{X_k\}$ $\lim\limits_{k \to \infty} \boldsymbol{f}(X_k) = A$
- Cauchy : $\lim\limits_{\Omega \ni X \to X_0} \boldsymbol{f}(X)$

$\forall \varepsilon > 0\ \exists \delta > 0\ \ \forall X' X'' \in \mathring{B}(X_0 \delta) \cap \Omega$
$\|\boldsymbol{f}(X') - \boldsymbol{f}(X'')\|_m < \varepsilon$

$$\lim_{(xy)\to(x_0y_0)} f(xy) \qquad \lim_{\substack{x\to x_0\\ y\to y_0}} f(xy) \qquad x_0 \;\; y_0$$

$$\lim_{\substack{x\to x_0\\ y\to\infty}} f(xy) \quad \lim_{\substack{x\to\infty\\ y\to\infty}} f(xy) \quad \lim_{\substack{x\to\infty\\ y\to+\infty}} f(xy)$$

( )

1 $\lim\limits_{(xy)\to(00)} \frac{\sin(x^2+y^2)}{x^2+y^2}$

: $\lim\limits_{(xy)\to(00)} \frac{\sin(x^2+y^2)}{x^2+y^2} \overset{\rho=\sqrt{x^2+y^2}}{=} \lim\limits_{\rho\to 0^+} \frac{\sin\rho^2}{\rho^2} = 1$

2 $\lim\limits_{(xy)\to(00)} \frac{xy}{\sqrt{x^2+y^2}}$

: $\forall (xy) \in \mathbb{R}^2 \setminus \{(00)\} \quad 0 \le \frac{|xy|}{\sqrt{x^2+y^2}} \le \frac{1}{2}\sqrt{x^2+y^2} \qquad \lim\limits_{(xy)\to(00)} \frac{xy}{\sqrt{x^2+y^2}} = 0$

3 $\lim\limits_{\substack{x\to\infty\\ y\to a}} \left(1-\frac{1}{2x}\right)^{\frac{x^2}{x+y}} \quad a\in\mathbb{R}$

$$: \lim_{\substack{x\to\infty\\ y\to a}} \ln\left(1-\frac{1}{2x}\right)^{\frac{x^2}{x+y}} = \lim_{\substack{x\to\infty\\ y\to a}} \frac{x^2}{x+y}\ln\left(1-\frac{1}{2x}\right)$$

$$= \lim_{\substack{x\to\infty\\ y\to a}} \frac{x^2}{x+y}\cdot\frac{-1}{2x} = \lim_{\substack{x\to\infty\\ y\to a}} \frac{-1}{2}\cdot\frac{1}{1+\frac{y}{x}} = -\frac{1}{2}$$

$\lim\limits_{\substack{x\to\infty\\ y\to a}} \left(1-\frac{1}{2x}\right)^{\frac{x^2}{x+y}} = e^{-\frac{1}{2}}$

4 $\lim\limits_{(xy)\to(00)} \frac{2xy}{x^2+y^2}$

: $\forall (xy) \in \mathbb{R}^2 \setminus \{(00)\} \quad g(xy) = \frac{2xy}{x^2+y^2}$ $\qquad A \; \forall kx \in \mathbb{R} \quad f_k(x) = (xkx) \quad \lim\limits_{x\to 0} f_k(x) = (00) \quad f_k \; \mathbb{R} \setminus \{0\} \quad (00) \qquad A = \lim\limits_{x\to 0} g(xkx) = \frac{2k}{1+k^2}$!

5 $f(xy) = \frac{x^2}{x^2+y^2-x}$ (xy) (00) 0 (xy) (00)

: $ab \in \mathbb{R}$ (00) $\begin{cases} x = at \\ y = bt \end{cases} (t \in \mathbb{R})$

$$\lim_{t \to 0} f(atbt) = \lim_{t \to 0} \frac{(at)^2}{(at)^2 + (bt)^2 - at} = \lim_{t \to 0} \frac{a^2}{a^2 + b^2 - \frac{a}{t}} = 0$$

$\forall t \in \mathbb{R} \quad g(t) = (t^2 t) \quad \lim_{t \to 0} g(t) = (00) \quad g\,\mathbb{R} \setminus \{0\} \quad (00)$

$$\lim_{t \to 0} f \circ g(t) = \lim_{t \to 0} \frac{(t^2)^2}{(t^2)^2 + t^2 - t^2} = 1 \neq 0$$

$$\lim_{(xy) \to (00)} f(xy)$$

6 $\lim\limits_{(xy)\to(00)} \frac{\sin(x^2+e^y)}{1+\ln(1+x)}$

: $\lim\limits_{(xy)\to(00)} \frac{\sin(x^2+e^y)}{1+\ln(1+x)} = \sin 1$

: $\lim\limits_{(xy)\to(x_0y_0)} f(xy)$

: $\lim\limits_{x\to x_0}\lim\limits_{y\to y_0} f(xy)$ $\lim\limits_{y\to y_0}\lim\limits_{x\to x_0} f(xy)$

: $\lim\limits_{x\to x_0}\lim\limits_{y\to y_0} f(xy)$ $x \neq x_0$ $\varphi(x) := \lim\limits_{y\to y_0} f(xy)$ $\lim\limits_{x\to x_0}\varphi(x)$

: ?

1:

7 $\lim_{(xy)\to(00)} \frac{2xy}{x^2+y^2}$ $y \neq 0$ $\lim_{x\to 0} \frac{2xy}{x^2+y^2} = 0$

$$\lim_{y\to 0} \lim_{x\to 0} \frac{2xy}{x^2+y^2} = 0$$

$\lim_{x\to 0} \lim_{y\to 0} \frac{2xy}{x^2+y^2} = 0$

2:

8 $\forall (xy) \in \mathbb{R}^2 (\ \ xy \neq 0)$

$$f(xy) = x \sin \frac{1}{y} + y \sin \frac{1}{x}$$

$\lim\limits_{x \to 0} x \sin \frac{1}{y} = 0$ $\quad \lim\limits_{x \to 0} y \sin \frac{1}{x}$ $\quad \lim\limits_{x \to 0} f(xy)$ $\quad \lim\limits_{y \to 0} f(xy)$ $\quad |f(xy)| \leq |x| + |y|$ $\quad \lim\limits_{(xy) \to (00)} f(xy) = 0$

1 $\lim\limits_{(xy)\to(x_0y_0)} f(xy) = A$ x_0 $U \lim\limits_{y\to y_0} f(xy) = \varphi(x)$

$$A = \lim_{x\to x_0} \varphi(x) = \lim_{x\to x_0} \lim_{y\to y_0} f(xy)$$

: $\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall (xy) \in \mathring{B}((x_0y_0)\delta) \ |f(xy) - A| < \varepsilon \ \forall x \in U \cap (x_0 - \delta x_0 + \delta)$ y $|\varphi(x) - A| \leq \varepsilon \ \lim\limits_{x\to x_0} \varphi(x) = A$

: $A \in \mathbb{R}$

1

: $\lim\limits_{(xy)\to(x_0y_0)} f(xy) = A$

$\lim\limits_{x\to x_0}\lim\limits_{y\to y_0} f(xy) = B \quad A = B$

2

: $\lim\limits_{x\to x_0}\lim\limits_{y\to y_0} f(xy) \quad \lim\limits_{y\to y_0}\lim\limits_{x\to x_0} f(xy)$

$\lim\limits_{(xy)\to(x_0y_0)} f(xy)$

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- : ; ;
- : $\mathbb{R}^m$ n
  m n

:

- $\lim\limits_{\Omega \ni X \to X_0} \boldsymbol{f}(X)$ $\lim\limits_{X \to X_0} \boldsymbol{f}(X)$
-
-
-
- Cauchy

:

:

- $\lim\limits_{(xy)\to(00)} \frac{\sin(x^2+y^2)}{x^2+y^2} = 1$
- $\lim\limits_{(xy)\to(00)} \frac{xy}{\sqrt{x^2+y^2}} = 0$
- $\lim\limits_{\substack{x\to\infty \\ y\to a}} \left(1-\frac{1}{2x}\right)^{\frac{x^2}{x+y}} = e^{-\frac{1}{2}} \quad a \in \mathbb{R}$
- $\lim\limits_{(xy)\to(00)} \frac{2xy}{x^2+y^2}$

:

: $\lim\limits_{(xy)\to(x_0y_0)} f(xy)$

: $\lim\limits_{x\to x_0}\lim\limits_{y\to y_0} f(xy)$ $\lim\limits_{y\to y_0}\lim\limits_{x\to x_0} f(xy)$

-
-
-

2 $mn \geq 1$ $\Omega \subseteq \mathbb{R}^n$ $X_0 \in \Omega$
Ω $\boldsymbol{f}\colon \Omega \to \mathbb{R}^m$

$$\lim_{\Omega \ni X \to X_0} \boldsymbol{f}(X) = \boldsymbol{f}(X_0)$$

$\boldsymbol{f}$ X_0

- $\boldsymbol{f}$ X_0 $\forall \varepsilon > 0\ \exists \delta > 0$
  $\forall X \in \Omega \quad \|X - X_0\|_n < \delta$

$$\|\boldsymbol{f}(X) - \boldsymbol{f}(X_0)\|_m < \varepsilon$$

  X_0 Ω
  $\boldsymbol{f}$ X_0
- $\boldsymbol{f}\,\Omega$ $\boldsymbol{f}\,\Omega$
- $\mathcal{C}(\Omega; \mathbb{R}^m) = \{\boldsymbol{f} \mid \boldsymbol{f} \colon \Omega \to \mathbb{R}^m \quad \}$
  $m = 1$ $\mathcal{C}(\Omega)$

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3

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4 $\Omega \subset \mathbb{R}^n$ $\boldsymbol{f}\colon \Omega \to \mathbb{R}^m$ $\boldsymbol{f}$ $\mathbb{R}^m$ G $\boldsymbol{f}^{-1}(G) = \{x \in \Omega \mid \boldsymbol{f}(x) \in G\}$

: $\mathbb{R}^m$ G $\boldsymbol{f}^{-1}(G)$ $X_0 \in \Omega$ $\forall \varepsilon > 0$ $G = B(\boldsymbol{f}(X_0)\varepsilon)$ $\boldsymbol{f}^{-1}(G)$ X_0 $\exists \delta > 0$ $B(X_0\delta) \subseteq \boldsymbol{f}^{-1}(G)$ $\forall X \in B(X_0\delta)$ $\|\boldsymbol{f}(X) - \boldsymbol{f}(X_0)\|_m < \varepsilon$ $\boldsymbol{f}$ X_0 $\boldsymbol{f}$

$\boldsymbol{f}$ $G\,\mathbb{R}^m$ $\forall X_0 \in \boldsymbol{f}^{-1}(G)$ $\boldsymbol{f}(X_0) \in G$ G $\exists \varepsilon > 0$ $B(\boldsymbol{f}(X_0)\varepsilon) \subseteq G$ $\boldsymbol{f}\,X_0$ $\exists \delta_1 > 0$ $\forall X \in \Omega \cap B(X_0\delta_1)$ $\|\boldsymbol{f}(X) - \boldsymbol{f}(X_0)\|_m < \varepsilon$ $\Omega \cap B(X_0\delta_1)$ $\exists \delta > 0$ $B(X_0\delta) \subseteq \Omega \cap B(X_0\delta_1)$ $\forall X \in B(X_0\delta)$ $\|\boldsymbol{f}(X) - \boldsymbol{f}(X_0)\|_m < \varepsilon$

$B(X_0\delta) \subseteq \boldsymbol{f}^{-1}\big(B(\boldsymbol{f}(X_0)\varepsilon\big) \subseteq \boldsymbol{f}^{-1}(G)$ X_0 $\boldsymbol{f}^{-1}(G)$ $\boldsymbol{f}^{-1}(G)$

3in

: $\boldsymbol{f}$ $\mathbb{R}^m$ G $\boldsymbol{f}^{-1}(G)$

5 ( ) $\Omega \subseteq \mathbb{R}^n$ $f \in \mathscr{C}(\Omega)$ f Ω

: f Ω $\forall k \in \mathbb{N}^*$ $\exists X_k \in \Omega$ $|f(X_k)| > k$ Ω $\{X_k\}$ $\{X_{\ell_k}\}$ A Ω $A \in \Omega$ f $f(A) = \lim_{k \to \infty} f(X_{\ell_k}) = \infty$! f

$f\ \Omega$ f f $-f$ $M = \sup f(\Omega)$ $\forall X \in \Omega\ f(X) < M$ $F(X) = \frac{1}{M - f(X)}$ $F \in \mathcal{C}(\Omega)$ M $\forall k \in \mathbb{N}^*\ \exists X_k \in \Omega\ f(X_k) > M - \frac{1}{k}\ F(X_k) > k$ $F\ \Omega$!

♣ :

1 $\lim\limits_{\Omega \ni X \to X_0} \boldsymbol{f}(X)$ $\lim\limits_{X \to X_0} \boldsymbol{f}(X)$

2

3

4

5 Cauchy

♣ :

: $\lim\limits_{(xy)\to(x_0y_0)} f(xy)$

: $\lim\limits_{x\to x_0}\lim\limits_{y\to y_0} f(xy)$ $\lim\limits_{y\to y_0}\lim\limits_{x\to x_0} f(xy)$

a)

b)

c)

♣

$mn \geq 1$ $\quad \Omega \subseteq \mathbb{R}^n$ $\quad X_0 \in \Omega$ $\quad \Omega$ $\quad \boldsymbol{f} \colon \Omega \to \mathbb{R}^m$

$$\lim_{\Omega \ni X \to X_0} \boldsymbol{f}(X) = \boldsymbol{f}(X_0)$$

$\boldsymbol{f}$ X_0

♣ :

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23in

$\Omega \subset \mathbb{R}^n$ $\boldsymbol{f}\colon \Omega \to \mathbb{R}^m$ $\boldsymbol{f}$ $\mathbb{R}^m$ G $\boldsymbol{f}^{-1}(G) = \{x \in \Omega \mid \boldsymbol{f}(x) \in G\}$

23in

( ) $\Omega \subseteq \mathbb{R}^n$ $f \in \mathcal{C}(\Omega)$ $f\,\Omega$

♣ : $\mathbb{R}^n$

1- $D \subseteq \mathbb{R}^n$ $\forall XY \in D$ D XY $\gamma : [01] \to D$ $\gamma(0) = X\, \gamma(1) = Y$

2-

1 ( ) $f \in \mathcal{C}(\Omega; \mathbb{R}^m)$ $\Omega \subseteq \mathbb{R}^n$ $f(\Omega)$

⋮

33in

2 ( ) $\Omega \subseteq \mathbb{R}^n$ $f \in \mathcal{C}(\Omega)$ $\forall X_1 X_2 \in \Omega$ $f(X_1)$ $f(X_2)$ μ $\exists X_0 \in \Omega$ $f(X_0) = \mu$

: 1 $f(\Omega)$ $\mathbb{R}$ $\forall X_1 X_2 \in \Omega$ $f(X_1) f(X_2)$ $\inf(\Omega)$ $f(\Omega)$

$\overset{1}{X} = \overset{:}{(x_1 \dots x_n)} \overset{mM}{} \in \mathbb{R}^n$

$$m \sum_{j=1}^{n} |x_j| \leq \|X\| \leq M \sum_{j=1}^{n} |x_j|$$

Analysis: X X $\frac{1}{M} \leq \sum_{j=1}^{n} \left| \frac{x_j}{\|X\|} \right| \leq \frac{1}{m}$ $y_j = \frac{x_j}{\|X\|}$ $Y = (y_1 \dots y_2)$ $\|Y\| = 1$

$$\frac{1}{M} \leq f(Y) := \sum_{j=1}^{n} |y_j| \leq \frac{1}{m}$$

f

: $S = \{ Y \in \mathbb{R}^n \mid \|Y\|_n = 1\}$ S $\forall Y = (y_1 \dots y_n) \in S$

$$f(Y) = \sum_{j=1}^{n} |y_j| > 0$$

f $a > 0$ b

$m = \frac{1}{b}$ $M = \frac{1}{a}$ $\forall X \in \mathbb{R}^n (X \quad)$ $Y = \frac{1}{\|X\|_n}(x_1 \dots x_n) \in S$ $a \leq f(Y) \leq b$ $a \leq \frac{1}{\|X\|_n} \sum_{j=1}^{n} |x_j| \leq b$

$$\frac{1}{b} \sum_{j=1}^{n} |x_j| \leq \|X\|_n \leq \frac{1}{a} \sum_{j=1}^{n} |x_j|$$

$$m \sum_{j=1}^{n} |x_j| \leq \|X\|_n \leq M \sum_{j=1}^{n} |x_j| \quad X$$

1 $n \geq 1$ $\Omega \subseteq \mathbb{R}^n$ $X_0 \in \mathbb{R}^n$ Ω $f : \Omega \to \mathbb{R}$

(1) $\lim\limits_{\Omega \ni X \to X_0} f(X) = 0$ f $\Omega \ni X \to X_0$

( )

$$f(X) = o(1) \ (\Omega \ni X \to X_0)$$

$\lim\limits_{\Omega \ni X \to X_0} f(X) = A$

$$f(X) - A = o(1) \ (\Omega \ni X \to X_0)$$

(2) $g : \Omega \to \mathbb{R}$ $\quad \beta > 0\ \delta > 0\ \ \forall X \in \Omega \cap \mathring{B}(X_0\delta)\ \ |f(X)| \leq \beta |g(X)|$

$$f(X) = O(g(X))\ (\Omega \ni X \to X_0)$$

$g(X) = O(f(X))$ $\quad fg$

(3) $k \geq 0$ $\quad \lim\limits_{\Omega \ni X \to X_0} \frac{f(X)}{\|X - X_0\|^k} = 0$ $\quad f\ \ \Omega \ni X \to X_0\ \ \|X - X_0\|^k$

$f(X) = o(\|X - X_0\|^k)\ (\Omega \ni X \to X_0)$

(4) $\lim_{\Omega \ni X \to X_0} \frac{f(X)}{\|X - X_0\|^k} = c \neq 0$ f $\Omega \ni X \to X_0$ $\|X - X_0\|$ k f $k = 0$ $\lim_{\Omega \ni X \to X_0} f(X) = c$ 0

!